Exact solution of a one-dimensional continuum percolation model

Alon Drory
Dipartimento di Fisica, Università La Sapienza, Piazzale Aldo Moro 2, Roma 00187, Italy
(Received 20 September 1996)

I consider a one-dimensional system of particles that interact through a hard core of diameter σ and can connect to each other if they are closer than a distance d. The mean cluster size increases as a function of the density ρ until it diverges at some critical density, the percolation threshold. This system can be mapped onto an off-lattice generalization of the Potts model, which I have called the Potts fluid, and in this way, the mean cluster size, pair connectedness, and percolation probability can be calculated exactly. The mean cluster size is $S=2\exp[\rho(d-\sigma)/(1-\rho\sigma)]-1$ and diverges only at the close-packing density $\rho_{\mathrm{CP}}=1/\sigma$. This is confirmed by the behavior of the percolation probability. These results should help in judging the effectiveness of approximations or simulation methods before they are applied to higher dimensions. [S1063-651X(97)05702-4]

PACS number(s): 64.60.Ak

I. INTRODUCTION

One-dimensional models have a long history in phase transition studies, going back to Ising's solution of the model that bears his name. Such models have been found to be free of phase transitions except in some singular circumstances (such as zero temperature), but nevertheless keep being studied because they are sometimes exactly solvable. Because of this, these models can serve as testing grounds, for example, for approximations that are then used in other dimensions. In this spirit, I present here the exact solution of a continuum percolation model in one dimension. Like related models, this model exhibits a phase transition only in some singular circumstances. However, it being exactly solvable may yet make it interesting.

In percolation on a lattice, the one-dimensional model is trivial. This is not the case for continuum percolation, where the objects connecting to each other may occupy arbitrary positions [1]. The richness of continuum percolation lies in the variety of the binding criterion (which may include effects such as the shape and spatial distribution of the objects) and the existence of interactions. The interplay between the connectivity criterion and the interactions makes the theory of continuum percolation a very challenging field. Therefore, even in one dimension, no completely general results can be derived for continuum percolation. However, some specific models can be solved. The main such model is the one-dimensional version of the extended spheres system used some time ago to model microemulsions [2].

The system consists of $N+1$ particles on a closed ring of length L. The particles interact with each other through a pure hard-core potential $v(x_i,x_j)\equiv v(i,j)$, i.e., for two particles i and j

$$v(x_i,x_j)=\begin{cases}\infty, & |x_i-x_j|<\sigma\\ 0, & |x_i-x_j|>\sigma,\end{cases} \tag{1.1}$$

where σ is the hard-core diameter.

The connectivity criterion is supplied by the existence of a soft (also called permeable) shell of diameter d around each particle. Two particles are bound if their shells overlap. Letting $p(x_i,x_j)\equiv p(i,j)$ be the probability that two particles at x_i and x_j are bound, we have

$$p(x_i,x_j)=\begin{cases}1, & |x_i-x_j|<d\\ 0, & |x_i-x_j|>d.\end{cases} \tag{1.2}$$

Naturally, $d>\sigma$. The percolation transition now arises as a function of the density $\rho=(N+1)/L$. The mean cluster size S diverges at a critical density ρ_c, which signals the sharp (in the thermodynamic limit) appearance of an infinite cluster. The order parameter is $P(\rho)$, the probability that a randomly selected particle belongs to this infinite cluster.

The solution of this model relies on a general mapping I described recently, between continuum percolation and a Potts fluid [3]. The Potts fluid is a system of freely moving spins $\{\lambda_i\}_{i=1}^{N}$ with s states, which interact with each other through a spin-dependent potential $V(x_i,\lambda_i;x_j,\lambda_j)$, such that

$$V(x_i,\lambda_i;x_j,\lambda_j)=\begin{cases}U(x_i,x_j) & \text{if } \lambda_i=\lambda_j\\ W(x_i,x_j) & \text{if } \lambda_i\neq\lambda_j.\end{cases} \tag{1.3}$$

Here U and W are arbitrary functions. Furthermore, the spins couple to an external field $h(x)$ through an interaction Hamiltonian

$$H_{\mathrm{int}}=-\sum_{i=1}^{N}\psi(\lambda_i)h(x_i), \tag{1.4}$$

where

$$\psi(\lambda)=\begin{cases}s-1 & \text{if } \lambda=1\\ -1 & \text{if } \lambda\neq 1.\end{cases} \tag{1.5}$$

Every continuum percolation model defined by an interaction $v(i,j)$ and a binding criterion $p(i,j)$ can be mapped exactly on a Potts fluid [3] by choosing

$$U(i,j)=v(i,j),$$
$$\exp[-\beta W(i,j)]=q(i,j)\exp[-\beta v(i,j)], \tag{1.6}$$

where

$$q(i,j)\equiv 1-p(i,j). \tag{1.7}$$

For a constant field, the Potts configuration integral is

$$Z=\frac{1}{N!}\sum_{\{\lambda_m\}}\int dx_1\cdots dx_N\exp\left[-\beta\sum_{i>j}V(i,j)+\beta h\sum_{i=1}^{N}\psi(\lambda_i)\right], \tag{1.8}$$

where the sum $\Sigma_{\{\lambda_m\}}$ is performed over all spin configurations and $\beta=1/kT$ as usual. The magnetization of the Potts fluid is defined as

$$M=\frac{1}{\beta N(s-1)}\frac{\partial \ln Z}{\partial h} \tag{1.9}$$

and the susceptibility is

$$\chi=\frac{\partial M}{\partial h}. \tag{1.10}$$

The Potts two-density function is defined as

$$\rho^{(2)}(\vec{x},\mu;\vec{y},\eta)=\left\langle\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\delta(\vec{x}_i-\vec{x})\,\delta(\vec{x}_j-\vec{y})\,\delta_{\lambda_i,\mu}\delta_{\lambda_j,\eta}\right\rangle. \tag{1.11}$$

According to the general mapping between continuum percolation and the Potts fluid, the percolation probability and the mean cluster size are obtained as the limits [3]

$$P(\rho)=\lim_{h\to 0}\lim_{N\to\infty}\lim_{s\to 1}M, \tag{1.12}$$

$$S(\rho)=\lim_{h\to 0}\lim_{N\to\infty}\lim_{s\to 1}\frac{1}{\beta}\chi \quad (\rho<\rho_c). \tag{1.13}$$

Hence the strategy is to calculate M and χ for the adequate Potts fluid and to obtain from them $P(\rho)$ and S by taking the appropriate limits. Furthermore, the Potts two-density function is related to a fundamental quantity in percolation, the pair-connectedness function $g^{\dagger}$, which is defined as

$$\rho^2 g^{\dagger}(\vec{x},\vec{y})d\vec{x}\,d\vec{y}=\mathcal{P}, \tag{1.14}$$

where $\mathcal{P}$ is the probability of finding two particles in regions $d\vec{x}$ and $d\vec{y}$ around the positions $\vec{x}$ and $\vec{y}$, such that they both belong to the same cluster. ρ is the numerical density. This function is related to the mean cluster size through the relation (for a translationally invariant system)

$$S=1+\frac{1}{N}\int d\vec{x}\,d\vec{y}\,\rho^2 g^{\dagger}(\vec{x},\vec{y}). \tag{1.15}$$

The relation to the two-density function is given by

$$g^{\dagger}(\vec{x},\vec{y})=\lim_{s\to 1}\frac{1}{\rho^2}[\rho^{(2)}(\vec{x},\mu;\vec{y},\mu)-\rho^{(2)}(\vec{x},\mu;\vec{y},\eta)] \quad (\rho<\rho_c), \tag{1.16}$$

where $\mu,\eta\neq 1$ and $\mu\neq\eta$, but are otherwise arbitrary.

For the percolation model defined by Eqs. (1.1) and (1.2), the proper Potts fluid is determined by Eq. (1.6) to be

$$\exp[-\beta U(x_i,x_j)]\equiv Q(i,j)=\begin{cases}0, & |x_i-x_j|<\sigma\\ 1, & |x_i-x_j|>\sigma,\end{cases}$$

$$\exp[-\beta W(x_i,x_j)]\equiv R(i,j)=\begin{cases}0, & |x_i-x_j|<d\\ 1, & |x_i-x_j|>d.\end{cases} \tag{1.17}$$

What makes this model solvable is that these functions can take only the values 0 and 1 (this condition may be relaxed if the values of d are restricted to $d<2\sigma$). Mathematically, this system presents several similarities with the Takahashi gas [4], and several parts of the following derivation follow the calculations of Takahashi and of Salsburg *et al.* [5].

II. SOLUTION OF THE MODEL

We assume that the spins are ordered on a closed ring of length L, so that the position 0 and the position L are identified. Along this ring, we place $N+1$ spins, numbered from 0 to N, so that one of the spins is fixed at the position 0. The property that makes such one-dimensional models solvable is the existence of a canonical ordering of the particles. For a configuration $\{x_0,x_1,\ldots,x_N\}$, the canonical ordering consists in labeling the leftmost particle (fixed at the position 0) as 0, the one immediately to its right as 1, and so on, until the rightmost particle, labeled N. Thus

$$0=x_0<x_1<x_2<\cdots<x_N<L\equiv x_{N+1}. \tag{2.1}$$

The definition $x_{N+1}\equiv L$ serves to simplify the notation later on. There are exactly $(N+1)!$ configurations that differ from the canonically ordered one only by the labels attached to the particles, provided we distinguish clockwise numbering from counterclockwise, i.e., provided the ring is oriented. Since all the position variables are integrated upon, each of these configurations contributes the same to Z, which is therefore $(N+1)!$ times the contribution of the canonically ordered configuration. Hence

$$Z=\int_0^L dx_N\int_0^{x_N}dx_{N-1}\cdots\int_0^{x_2}dx_1\sum_{\{\lambda_m\}}\exp\left[-\beta\sum_{i>j}V(i,j)+\beta h\sum_{i=0}^{N}\psi(\lambda_i)\right]. \tag{2.2}$$

This property is completely general and is independent of the specific form of $V(i,j)$. Let us denote

$$f_{\lambda_i,\lambda_j}(|x_i-x_j|)\equiv\exp[-\beta V(x_i,x_j)]=\begin{cases}Q(i,j), & \lambda_i=\lambda_j\\ R(i,j), & \lambda_i\neq\lambda_j.\end{cases} \tag{2.3}$$

Now, taking into account that $x_0=x_{N+1}$ because of the periodic boundary conditions, the configuration integral can be written as

$$Z=\int_0^L dx_N\int_0^{x_N} dx_{N-1}\cdots\int_0^{x_2} dx_1 \times\sum_{\{\lambda_m\}}\prod_{i>j}^{N} f_{\lambda_i,\lambda_j}(|x_i-x_j|)\exp\left[\beta h\sum_{i=0}^{N}\psi(\lambda_i)\right]. \tag{2.4}$$

The specifics of the model, Eq. (1.17), now enter to verify the following theorem.

Theorem (factorization). For any spin configuration $\{\lambda_m\}$,

$$\prod_{i>j=0}^{N} f_{\lambda_i,\lambda_j}(|x_i-x_j|)=\prod_{i=0}^{N-1} f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i). \tag{2.5}$$

Although simple, this theorem is not entirely trivial because of the presence of two scales σ and d, the interplay of which depends on the spin configuration $\{\lambda_m\}$. The theorem is proved in the Appendix. Using this result, Eq. (2.4) can now be rewritten as

$$Z=\int_0^L dx_N\int_0^{x_N} dx_{N-1}\cdots\int_0^{x_2} dx_1\sum_{\{\lambda_m\}}\left\{f_{\lambda_0,\lambda_N}(L-x_N) \times\prod_{i=1}^{N} f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i)f_{\lambda_1,\lambda_0}(x_1) \times\exp\left[\beta h\sum_{i=1}^{N}\psi(\lambda_i)\right]\right\}. \tag{2.6}$$

Let us define

$$G_{\lambda_i,\lambda_j}(y)\equiv f_{\lambda_i,\lambda_j}(y)\exp\left\{\frac{\beta h}{2}[\psi(\lambda_i)+\psi(\lambda_j)]\right\}. \tag{2.7}$$

Then

$$Z=\int_0^L dx_N\cdots\int_0^{x_2} dx_1\sum_{\{\lambda_m\}}\{G_{\lambda_0,\lambda_N}(L-x_N) \times G_{\lambda_N,\lambda_{N-1}}(x_N-x_{N-1})\cdots G_{\lambda_2,\lambda_1}(x_2-x_1)G_{\lambda_1,\lambda_0}(x_1)\}. \tag{2.8}$$

This multiple integral has the form of a Laplace convolution. Denoting

$$T_{\lambda_i,\lambda_j}(\omega)\equiv\int_0^\infty dy\, e^{-\omega y}G_{\lambda_i,\lambda_j}(y), \tag{2.9}$$

we have

$$\begin{aligned}\int_0^\infty dL\, e^{-\omega L}Z(L)&=\sum_{\{\lambda_0,\lambda_1,\ldots,\lambda_N\}} T_{\lambda_0,\lambda_N}(\omega) \times T_{\lambda_N,\lambda_{N-1}}(\omega)\cdots T_{\lambda_2,\lambda_1}(\omega)T_{\lambda_1,\lambda_0}(\omega)\\ &=\mathrm{Tr}(T^{N+1}).\end{aligned} \tag{2.10}$$

The matrix $T_{\lambda_i,\lambda_j}(\omega)$ now plays the role of an effective transfer matrix for this problem. It is an $s\times s$ matrix of the form

$$T=\begin{pmatrix} A & B & B & \cdots & B\\ B & C & D & \cdots & D\\ B & D & C & \cdots & D\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ B & D & D & \cdots & C\end{pmatrix}, \tag{2.11}$$

where

$$\begin{aligned}A&=\int_0^\infty dy\, e^{-\omega y}Q(y)e^{\beta(s-1)h}=\frac{1}{\omega}\exp[-\omega\sigma+\beta(s-1)h],\\ B&=\int_0^\infty dy\, e^{-\omega y}R(y)e^{(1/2)\beta(s-2)h}\\ &=\frac{1}{\omega}\exp\left[-\omega d+\frac{1}{2}\beta(s-2)h\right],\\ C&=\int_0^\infty dy\, e^{-\omega y}Q(y)e^{-\beta h}=\frac{1}{\omega}\exp[-\omega\sigma-\beta h],\\ D&=\int_0^\infty dy\, e^{-\omega y}R(y)e^{-\beta h}=\frac{1}{\omega}\exp[-\omega d-\beta h].\end{aligned} \tag{2.12}$$

Let $\{\alpha_i\}_{i=1}^s$ be the eigenvalues of T. Then

$$\int_0^\infty dL\, e^{-\omega L}Z(L)=\sum_{i=1}^{s}\alpha_i^{N+1}. \tag{2.13}$$

The eigenvalues are found easily to be

$$\begin{aligned}\alpha_1&=\frac{1}{2}[A+C+(s-2)D+\sqrt{\Delta}],\\ \alpha_2&=\frac{1}{2}[A+C+(s-2)D-\sqrt{\Delta}],\\ \alpha_3&=\cdots=\alpha_s=C-D,\end{aligned} \tag{2.14}$$

where

$$\Delta=[A-C-(s-2)D]^2+4(s-1)B^2. \tag{2.15}$$

Reversing the Laplace transform in Eq. (2.13) yields

$$Z=\sum_{j=1}^{s}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\omega\exp[\omega L+(N+1)\ln\alpha_j(\omega)]. \tag{2.16}$$

In the thermodynamic limit, the integral can be evaluated *exactly* by steepest descent. Let us introduce the notation

$$\gamma_j(\omega)\equiv\omega\frac{L}{N+1}+\ln\alpha_j(\omega). \tag{2.17}$$

Then the maximum of the exponents in Eq. (2.16) is obtained from the condition

$$\left.\frac{d\gamma_j}{d\omega}\right|_{\omega=\omega_j}=0, \tag{2.18}$$

which defines the quantity ω_j at which $\gamma_j(\omega)$ is maximal. From the usual theory of steepest descent [6], we now obtain from Eq. (2.16) that

$$Z=\sum_{j=1}^{s} K_j \exp[(N+1)\gamma_j(\omega_j)], \tag{2.19}$$

where

$$K_j=\frac{1}{[2\pi(N+1)\gamma_j''(\omega_j)]^{1/2}}, \quad \gamma_j''(\omega)=\frac{d^2\gamma_j(\omega)}{d\omega^2}. \tag{2.20}$$

The sum in Eq. (2.19) is dominated by γ_1. To see this, note first that since $\Delta>0$, we have $\alpha_1>\alpha_2$. Furthermore,

$$\alpha_1-\alpha_3=\frac{1}{2}[A-C+sD+\sqrt{\Delta}]. \tag{2.21}$$

Since $A=Ce^{s\beta h}$ and $h>0$, we have that $A-C>0$. Therefore, $\alpha_1-\alpha_3>0$. Hence, for any given ω, $\alpha_1(\omega)>\alpha_j(\omega)$ for $j=2,\ldots,s$. Since $\gamma_1(\omega)-\gamma_j(\omega)=\ln(\alpha_1/\alpha_j)$, we also have that for any given ω, $\gamma_1(\omega)>\gamma_j(\omega)$ for $j=2,\ldots,s$. Now, at ω_1, γ_1 is maximal, so that

$$\gamma_1(\omega_1)\geqslant\gamma_1(\omega_j)>\gamma_j(\omega_j) \tag{2.22}$$

for $j=2,\ldots,s$. Hence, for $N\to\infty$,

$$Z=K_1e^{N\gamma_1}\left\{1+\sum_{j=2}^{s}\frac{K_j}{K_1}\exp[N\gamma_1(\omega_1)-N\gamma_j(\omega_j)]\right\} \sim K_1e^{N\gamma_1}. \tag{2.23}$$

Hence

$$\lim_{N\to\infty}\frac{1}{N}\ln Z=\gamma_1. \tag{2.24}$$

From Eq. (2.24), we can obtain the magnetization M,

$$\begin{aligned} M&=\frac{1}{\beta(s-1)N}\frac{d\ \ln Z}{dh}\\ &=\frac{1}{\beta(s-1)N}\left[\left(\frac{\partial\gamma_1}{\partial h}\right)_{\omega_1}+\left(\frac{\partial\gamma_1}{\partial\omega_1}\right)_h\left(\frac{\partial\omega_1}{\partial h}\right)\right]\\ &=\frac{1}{\beta(s-1)}\left(\frac{\partial\gamma_1}{\partial h}\right)_{\omega_1}, \end{aligned} \tag{2.25}$$

where we made use of the fact that $0=(\partial\gamma_1/\partial\omega_1)_h$ by definition of ω_1.

Although Eq. (2.24) is in principle the exact solution of the one-dimensional Potts fluid, it is not explicit enough to be useful. However, we are not interested in the Potts fluid itself, but rather in the percolation model. This is obtained in the limits $s\to1$ and $h\to0$, and in these limits all quantities can be calculated explicitly.

To do this, we set $s=1+\epsilon$ and calculate all relevant quantities to first order only in ϵ (this turns out to be sufficient). Therefore, from Eqs. (2.12) and (2.17), we find after some algebra that

$$\gamma_1=\omega\frac{L}{N}+\ln A+\epsilon\left[\beta h+\frac{B^2}{A(A-C+D)}\right]+O(\epsilon^2), \tag{2.26}$$

where A,B,C,D must be evaluated to zeroth order in ϵ. From Eq. (2.12), the result for this is

$$A=\frac{1}{\omega}\exp[-\omega\sigma], \quad B=\frac{1}{\omega}\exp\left[-\omega d-\frac{1}{2}\beta h\right],$$
$$C=\frac{1}{\omega}\exp[-\omega\sigma-\beta h], \quad D=\frac{1}{\omega}\exp[-\omega d-\beta h]. \tag{2.27}$$

Now

$$M=\lim_{\epsilon\to0}\frac{1}{\beta\epsilon}\left(\frac{\partial\gamma_1}{\partial h}\right)_{\omega_1}=1+\frac{1}{\beta}\frac{\partial}{\partial h}\left[\frac{B^2}{A(A-C+D)}\right]_{\omega_1,\epsilon=0}. \tag{2.28}$$

After some algebra, and with the use of Eq. (2.27), we end up with

$$M=1-\left[\frac{B}{A-C+D}\right]^2_{\omega_1,\epsilon=0}. \tag{2.29}$$

After some more algebra, the susceptibility is obtained straightforwardly,

$$\chi=\frac{\partial M}{\partial h}=\beta\left[\frac{B^2(A+C-D)}{(A-C+D)^2}\right]_{\omega_1,\epsilon=0}. \tag{2.30}$$

In both Eqs. (2.29) and (2.30), ω_1 is evaluated at $\epsilon=0$. From Eqs. (2.26) and (2.27), we have that for $\epsilon=0$,

$$\gamma_1=\frac{\omega}{\rho}-\omega\sigma-\ln\omega, \tag{2.31}$$

where $\rho=(N+1)/L$ is the density. Hence, from $(d\gamma_1/d\omega)_{\omega=\omega_1}=0$, we obtain

$$\omega_1|_{\epsilon=0}=\frac{\rho}{1-\rho\sigma}. \tag{2.32}$$

The final result is obtained from Eqs. (2.29) and (2.30) by setting $\epsilon=0$, plugging in the value of ω_1, and taking the limit $h\to0$. We have

$$M\to P(\rho)=1-\lim_{h\to0}\left\{1-(e^{\beta h}-1)\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]\right\}^{-2}, \tag{2.33}$$

$$\frac{\chi}{\beta}\to S(\rho)=2\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]-1. \tag{2.34}$$

If we just set $h=0$ in the equation for $P(\rho)$, we obtain $P(\rho)=0$, which shows that there is no transition. There is, however, one exception, obtained at $\rho=1/\sigma=\rho_{\mathrm{CP}}$. This is the close-packing density in one dimension, and clearly we must have a percolating cluster in this case. Indeed, at this density

$$\lim_{\rho\to\rho_{\rm CP}} = \exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]=\infty, \tag{2.35}$$

and therefore

$$\left\{1-(e^{\beta h}-1)\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]\right\}^{-2}\Bigg|_{\rho=\rho_{\rm CP}} = 0 \tag{2.36}$$

for *all* values of h; hence $P(\rho_{\rm CP})=1$, as expected. Note that for $\sigma=0$ (no hard core), $\rho_{\rm CP}\to\infty$. Equation (2.33) becomes, in this case,

$$P(\rho)=1-\lim_{h\to 0}[1-(e^{\beta h}-1)e^{\rho d}]^{-2}, \tag{2.37}$$

and indeed $P(\rho=\infty)=1$. The transition is clearer in the expression Eq. (2.34) for the mean cluster size

$$\lim_{\rho\to\rho_{\rm CP}} S(\rho)=\lim_{\rho\to\rho_{\rm CP}} 2\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]-1=\infty, \tag{2.38}$$

so that indeed the mean cluster size diverges as the density approaches the close-packing value.

III. THE PAIR-CONNECTEDNESS FUNCTION

Let us now consider the pair connectedness. To this end we need to calculate the spin two-density function

$$\rho^{(2)}(x,\mu;y,\eta)=\left\langle \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \delta(x_i-x)\,\delta(x_j-y)\,\delta_{\lambda_i,\mu}\delta_{\lambda_j,\eta}\right\rangle. \tag{3.1}$$

To simplify the algebra let us use the system's translational invariance [one can work with Eq. (3.1) and obtain translational invariance directly, but this adds needlessly to the calculations]. We assume that $y<x$, denote $x-y=r$, and set $y=0$ and $\lambda_0=\mu$. We now have, therefore, that

$$\begin{aligned}\rho^{(2)}(0,\mu;r,\eta)&\equiv\rho^{(2)}(r;\mu,\eta)\\ &=\rho\left\langle\sum_{k=1}^{N}\delta(x_k-r)\,\delta_{\lambda_0,\mu}\delta_{\lambda_k,\eta}\right\rangle,\end{aligned} \tag{3.2}$$

where $\rho=(N+1)/L$. From Eq. (2.8), we have that

$$\begin{aligned}&\rho^{(2)}(r;\mu,\eta)\\ &=\frac{\rho}{Z}\sum_{k=1}^{N}\int_0^L dx_N\cdots\int_0^{x_2}dx_1\\ &\times\sum_{\{\lambda_m\}}\{\delta_{\lambda_0,\mu}G_{\lambda_0,\lambda_N}(L-x_N)\\ &\times G_{\lambda_N,\lambda_{N-1}}(x_N-x_{N-1})\cdots G_{\lambda_{k+1},\lambda_k}(x_{k+1}-x_k)\\ &\times\delta_{\lambda_k,\eta}\delta(x_k-r)G_{\lambda_k,\lambda_{k-1}}(x_k-x_{k-1})\cdots G_{\lambda_1,\lambda_0}(x_1)\},\end{aligned} \tag{3.3}$$

where the function $G_{\lambda_i,\lambda_j}(x)$ is defined in Eq. (2.7). This is again a Laplace convolution and we have

$$\begin{aligned}&\int_0^\infty dL\,e^{-\omega L}\frac{Z}{\rho}\rho^{(2)}(r;\mu,\eta)\\ &=\sum_{k=1}^{N}\sum_{\{\lambda_0,\lambda_1,\ldots,\lambda_N\}}\delta_{\lambda_0,\mu}T_{\lambda_0,\lambda_N}(\omega)\cdots T_{\lambda_{k+1},\lambda_k}(\omega)\\ &\times\delta_{\lambda_k,\eta}e^{-\omega r}h_k(r;\lambda_k,\lambda_0),\end{aligned} \tag{3.4}$$

where $h_k(r;\lambda_k,\lambda_0)$ is defined by its Laplace transform as

$$\begin{aligned}&\int_0^\infty dr\,e^{-\nu r}h_k(r;\lambda_k,\lambda_0)\\ &=\sum_{\{\lambda_{k-1},\ldots,\lambda_1\}}T_{\lambda_k,\lambda_{k-1}}(\nu)\cdots T_{\lambda_1,\lambda_0}(\nu)\\ &=(T^k)_{\lambda_k,\lambda_0}.\end{aligned} \tag{3.5}$$

The sum over the spins yields

$$\begin{aligned}\int_0^\infty dL\,e^{-\omega L}\frac{Z}{\rho}\rho^{(2)}(r;\mu,\eta)&=\sum_{k=1}^{N}[T^{N-k+1}(\omega)]_{\mu,\eta}\\ &\times e^{-\omega r}h_k(r;\eta,\mu).\end{aligned} \tag{3.6}$$

Inverting the Laplace transform in Eqs. (3.5) and (3.6), we obtain finally

$$\begin{aligned}\frac{Z}{\rho}\rho^{(2)}(r;\mu,\eta)&=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\omega\,\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\nu\\ &\times\sum_{k=1}^{N}[T^{N-k+1}(\omega)]_{\mu,\eta}[T^k(\nu)]_{\eta,\mu}\\ &\times e^{\omega(L-r)}e^{\nu r}.\end{aligned} \tag{3.7}$$

Let us introduce now a basis of eigenvectors of the matrix $T(\zeta)$, which will be denoted $\{\langle u_1|,\langle u_2|,\ldots,\langle u_s|\}$, $\langle u_j|$ being associated with the eigenvalue α_j defined in Eq. (2.14). These can be calculated straightforwardly. Furthermore, the resulting expressions can be simplified by setting the magnetic field $h=0$, since it plays no part in the calculation of the pair connectedness. With this, we have that $A=C$ and $B=D$. From Eqs. (2.12) and (2.14), we now have that

$$\alpha_1=A+(s-1)B,\quad \alpha_2=A-B,$$
$$A(\zeta)=\frac{1}{\zeta}\exp(-\zeta\sigma),\quad B(\zeta)=\frac{1}{\zeta}\exp(-\zeta d). \tag{3.8}$$

The eigenvectors are now found to be

$$\langle u_1|=\frac{1}{\sqrt{s}}(1,1,1,\ldots,1), \tag{3.9}$$

$$\langle u_2|=\frac{1}{\sqrt{s(s-1)}}(s-1,-1,-1,\ldots,-1),$$

$$\langle u_3|=\frac{1}{\sqrt{2}}(0,1,-1,0,\ldots,0),$$

$$\vdots$$

$$\langle u_s|=\frac{1}{\sqrt{(s-1)(s-2)}}(0,1,1,\ldots,-(s-2)).$$

Finally, we define two vectors of the type (0,0, . . . ,1,0, . . . ,0), denoted $\langle\mu|$ and $\langle\eta|$ such that their single nonzero component 1 stands at the position μ and η, respectively. With this, we have that

$$[T^{N-k+1}(\omega)]_{\mu,\eta}=\sum_{m=1}^{s}\langle\mu|u_m\rangle\alpha_m^{N-k+1}(\omega)\langle u_m|\eta\rangle,$$

$$[T^k(\nu)]_{\eta,\mu}=\sum_{n=1}^{s}\langle\eta|u_n\rangle\alpha_n^k(\nu)\langle u_n|\mu\rangle. \tag{3.10}$$

Hence

$$\begin{aligned}&\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\omega[T^{N-k+1}(\omega)]_{\mu,\eta}e^{\omega(L-r)}\\&=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\omega\,e^{\omega L}\sum_{m=1}^{s}\alpha_m^{N+1}(\omega)F_{m,k}(\omega,r)\\&=\sum_{m=1}^{s}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\omega\,e^{(N+1)\gamma_m(\omega)}F_{m,k}(\omega,r),\end{aligned} \tag{3.11}$$

where

$$F_{m,k}(\omega,r)=\frac{\langle\mu|u_m\rangle\langle u_m|\eta\rangle}{[\alpha_m(\omega)]^k}e^{-\omega r}, \tag{3.12}$$

and $\gamma_m(\omega)=\omega L/(N+1)+\ln\alpha_m(\omega)$ as in Eq. (2.17). In the limit $N,L\to\infty$ the integral can be evaluated exactly by steepest descent. The calculation is identical to the one performed in Eqs. (2.16)–(2.19). As in Eq. (2.24), the term γ_1 dominates all the others, so that in the limit $N\to\infty$, the result is exactly

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\omega[T^{N-k+1}(\omega)]_{\mu,\eta}e^{\omega(L-r)}\\&\quad=\lim_{N\to\infty}K_1\exp[(N+1)\gamma_1(\omega_1)]F_{1,k}(\omega_1,r).\end{aligned} \tag{3.13}$$

From Eq. (2.24), we have that, in the same limit, $Z=K_1\exp[(N+1)\gamma_1(\omega_1)]$. Therefore, combining Eqs. (3.7), (3.10), and (3.13), we finally obtain the expression

$$\begin{aligned}\rho^{(2)}(r;\mu,\eta)&=\lim_{N\to\infty}\rho\sum_{k=1}^{N}F_{1,k}(\omega_1,r)\frac{1}{2\pi i}\\&\quad\times\int_{a-i\infty}^{a+i\infty}d\nu[T^k(\nu)]_{\eta,\mu}e^{\nu r}\\&=\rho\sum_{k=1}^{\infty}\frac{\langle\mu|u_1\rangle\langle u_1|\eta\rangle}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\nu\sum_{n=1}^{s}\langle\eta|u_n\rangle\\&\quad\times\langle u_n|\mu\rangle\left[\frac{\alpha_n(\nu)}{\alpha_1(\omega_1)}\right]^k e^{(\nu-\omega_1)r}.\end{aligned} \tag{3.14}$$

In order to calculate the pair connectedness, we need the case $\mu,\eta\neq1$ [see Eq. (1.16)]. For this case, we see directly from the definition of the $\langle u_i|$, Eq. (3.9), that for any $\lambda\geqslant2$,

$$\langle u_1|\lambda\rangle=\frac{1}{\sqrt{s}},\quad\langle u_2|\lambda\rangle=\frac{-1}{\sqrt{s(s-1)}}, \tag{3.15}$$

and for any $n\geqslant3$,

$$\langle u_n|\lambda\rangle=\begin{cases}0 & \text{if } n<\lambda\\ \dfrac{-(n-2)}{\sqrt{(n-2)(n-1)}} & \text{if } n=\lambda\\ \dfrac{1}{\sqrt{(n-2)(n-1)}} & \text{if } n>\lambda.\end{cases} \tag{3.16}$$

We are interested, however, in the pair connectedness, which involves the difference $\rho^{(2)}(r;\mu,\mu)-\rho^{(2)}(r;\mu,\eta)$. Substituting Eq. (3.15) into Eq. (3.14), we obtain therefore

$$\begin{aligned}&\rho^{(2)}(r;\mu,\mu)-\rho^{(2)}(r;\mu,\eta)\\&\quad=\frac{\rho}{s}\sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\nu\sum_{n=1}^{s}[\langle\mu|u_n\rangle-\langle\eta|u_n\rangle]\langle u_n|\mu\rangle\\&\qquad\times\left[\frac{\alpha_n(\nu)}{\alpha_1(\omega_1)}\right]^k e^{(\nu-\omega_1)r}.\end{aligned} \tag{3.17}$$

From Eq. (3.15), we see that $\langle\mu|u_n\rangle-\langle\eta|u_n\rangle$ vanishes identically for $n=1,2$. Now, for $n\geqslant3$, we have that

$$\alpha_n(\nu)=\frac{1}{\nu}[e^{-\nu\sigma}-e^{-\nu d}] \tag{3.18}$$

and is independent of the value of n. Finally, from Eq. (3.16), it is easily seen that for any $\mu,\eta\neq1$, $\mu\neq\eta$, we have that

$$\sum_{n=3}^{s}\langle u_n|\mu\rangle[\langle\mu|u_n\rangle-\langle\eta|u_n\rangle]=1. \tag{3.19}$$

Therefore, we have that

$$\begin{aligned}&\rho^{(2)}(r;\mu,\mu)-\rho^{(2)}(r;\mu,\eta)\\&\quad=\frac{\rho}{s}\sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\nu\left[\frac{\alpha_n(\nu)}{\alpha_1(\omega_1)}\right]^k e^{(\nu-\omega_1)r}.\end{aligned} \tag{3.20}$$

Finally, to obtain $g^+(r)$, we take the limit $s\to1$. In this limit, Eq. (3.8) yields

$$\alpha_1(\omega_1)=\frac{1}{\omega_1}\exp[-\omega_1\sigma], \tag{3.21}$$

where, from Eq. (2.32),

$$\omega_1=\frac{\rho}{1-\rho\sigma}. \tag{3.22}$$

Hence, substituting the values of α_n and α_1, we finally obtain, by substituting Eq. (3.20) into Eq. (1.16), that

$$\begin{aligned}g^+(r)&=\lim_{s\to1}\frac{1}{\rho^2}[\rho^{(2)}(r;\mu,\mu)-\rho^{(2)}(r;\mu,\eta)]\\&=\frac{1}{\rho}\sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\nu\frac{\omega_1^k}{\nu^k}[1-e^{-\nu(d-\sigma)}]^k e^{(\nu-\omega_1)r}\end{aligned} \tag{3.23}$$

or

$$g^{\dagger}(r)=\frac{1}{\rho}\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\omega_1^k e^{-\omega_1(r-k\sigma)}\frac{1}{2\pi i}\times\int_{a-i\infty}^{a+i\infty}d\nu\,\frac{1}{\nu^k}e^{\nu[r-k\sigma-j(d-\sigma)]}. \tag{3.24}$$

This integral is an immediate inverse Laplace transform. The final result is therefore

$$g^{\dagger}(r)=\frac{1}{\rho}\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\times\left(\frac{\rho}{1-\rho\sigma}\right)^k\frac{[r-k\sigma-j(d-\sigma)]^{k-1}}{(k-1)!}\times\Theta[r-k\sigma+j(d-\sigma)]\exp\left[-\frac{\rho(r-k\sigma)}{1-\rho\sigma}\right], \tag{3.25}$$

where $\Theta(z)$ is the step function

$$\Theta(z)=\begin{cases}1, & z>0\\ 0, & z<0.\end{cases} \tag{3.26}$$

We can understand the meaning of k and j as follows. Given a distance $r>0$ from the (arbitrary) origin, $g^{\dagger}(r)$ is a sum of contributions from sets of configurations indexed by k. The derivation shows that in each such configuration there are exactly k particles within the interval r. Since the potential includes a hard core of diameter σ, we expect a condition on k such that $r>k\sigma$. This is indeed contained in the expression $\Theta[r-k\sigma+j(d-\sigma)]$, since it implies that $r>k\sigma+j(d-\sigma)>k\sigma$. Next, we note that each set of configurations containing k particles in the interval r is further divided into subsets indexed by j. The meaning of j is made clear from the condition $r>k\sigma+j(d-\sigma)=jd+(k-j)\sigma$ generated by the step function. This implies that j particles out of the k are singlets, i.e., are *not* connected to any other particle in the set. Therefore, the step function shows that $g^{\dagger}(r)$ is a sum of separate contributions from all sets of configurations containing k particles in the interval r,j of which are not connected to any other particle in the set.

Finally, we can check the expected relation between the mean cluster size S and the pair connectedness. This is worked out most conveniently from Eq. (3.23). From this, we have that

$$\int_0^{\infty}dr\,g^{\dagger}(r)=\frac{1}{\rho}\sum_{k=1}^{\infty}\left\{\frac{\omega_1^k}{\nu^k}[1-e^{-\nu(d-\sigma)}e^{(\nu-\omega_1)r}]^k\right\}_{\nu=\omega_1}=\frac{1}{\rho}\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]-1. \tag{3.27}$$

From Eq. (1.15), and remembering that we have calculated $g^{\dagger}(x,y)$ under the assumption that $y<x$, we now have that

$$S=1+\frac{1}{N}\int dx\,dy\,\rho^2 g^{\dagger}(x,y)=1+2\rho\int_0^{\infty}dr\,g^{\dagger}(r) \tag{3.28}$$

and therefore

$$S=2\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]-1, \tag{3.29}$$

in agreement with the previously obtained result, Eq. (2.34).

IV. SUMMARY

I have obtained the exact solution of a one-dimensional percolation model that includes the effects of interactions among the particles, modeled by a hard core of diameter σ. The particles are connected to each other if their centers are closer than a distance d. The clustering depends on the density ρ of the particles. By mapping this system on a Potts fluid, one can calculate the mean cluster size, pair connectedness, and percolation probability of this system. As with other one-dimensional systems, the phase transition occurs only in some extreme circumstance, in this case, only when the density reaches the close-packing value $\rho_{\mathrm{CP}}=1/\sigma$. As the density increases towards this critical value, the mean cluster size increases as

$$S=2\exp\left[\frac{\rho(d-\sigma)}{1-\rho\sigma}\right]-1 \tag{4.1}$$

and diverges at ρ_{CP}. Meanwhile, the pair connectedness is given by

$$g^{\dagger}(r)=\frac{1}{\rho}\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\times\left(\frac{\rho}{1-\rho\sigma}\right)^k\frac{[r-k\sigma-j(d-\sigma)]^{k-1}}{(k-1)!}\times\Theta[r-k\sigma+j(d-\sigma)]\exp\left[-\frac{\rho(r-k\sigma)}{1-\rho\sigma}\right]. \tag{4.2}$$

These results should be particularly useful as a test of approximation methods or numerical calculations or simulations. Such methods can be checked against the exact results and some idea thus obtained of their reliability and effectiveness. Once a method has proved to be successful in the one-dimensional case, it can be applied to higher dimensions with some hope of success.

From a more general point of view, the present results are a confirmation of the power provided by the mapping between continuum percolation and the Potts fluid. Recently, Cinlar and Torquato have used renewal theory to investigate one-dimensional continuum percolation, but without interactions (i.e., the $\sigma\to 0$ limit of the present model) [7]. There seems to be no simple way of extending this type of direct probability arguments to cover interactions as well. It appears that in order to discuss general models, which include inter-particle interactions, there is at the moment no method more powerful than the mapping with the Potts fluid.

APPENDIX

The proof of the factorization theorem is obtained by induction on N. For $N=1$ the theorem is obvious. Let us assume it is correct for N and prove it for $N+1$. Then

$$\prod_{i>j=0}^{N+1} f_{\lambda_i,\lambda_j}(|x_i-x_j|)$$
$$= \prod_{i>j=0}^{N} f_{\lambda_i,\lambda_j}(|x_i-x_j|) \prod_{i=0}^{N} f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)$$
$$= \prod_{i=0}^{N-1} f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) \prod_{i=0}^{N} f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)$$
$$= \prod_{i=0}^{N-1} [f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)]$$
$$\times f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N), \tag{A1}$$

where we used the induction assumption to obtain the second equality.

Let us now consider the various possibilities for every term i.

(i) $f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)=1$. In this case clearly,

$$f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i) = f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i). \tag{A2}$$

(ii) $f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)=0$ *and* $\lambda_{N+1}=\lambda_i$. Then, from Eq. (2.3),

$$f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i) = Q(x_{N+1}-x_i) = 0, \tag{A3}$$

so that, necessarily [see Eq. (1.1)], $x_{N+1}-x_i<\sigma$. However, in the canonical ordering, $x_i<x_{i+1}<\cdots<x_N$, so that

$$x_{N+1}-x_N<x_{N+1}-x_i<\sigma<d. \tag{A4}$$

Hence

$$f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N)=0, \tag{A5}$$

even if $\lambda_{N+1}\neq\lambda_N$, because $\sigma<d$. Therefore,

$$0 = f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i) \times f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N)$$
$$= f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N). \tag{A6}$$

(iii) $f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)=0$ *and* $\lambda_{N+1}\neq\lambda_i$. Then [see Eq. (2.3)],

$$f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i) = R(x_{N+1}-x_i) = 0 \tag{A7}$$

and, necessarily [see Eq. (1.1)], $x_{N+1}-x_i<d$. Then there must exist some j, $i\leq j<N+1$, such that

$$\lambda_{N+1}=\lambda_k, \quad \lambda_{N+1}\neq\lambda_j \quad \text{for all } j<k<N+1, \tag{A8}$$

i.e., j is the closest spin to λ_{N+1}, which differs from it. In particular, $\lambda_{N+1}=\lambda_{j+1}$, so that

$$f_{\lambda_{j+1},\lambda_j}(x_{j+1}-x_j) = R(x_{j+1}-x_j). \tag{A9}$$

Because of the canonical ordering,

$$x_{j+1}-x_j\leq x_{j+1}-x_i<x_{N+1}-x_i<d. \tag{A10}$$

Hence

$$R(x_{j+1}-x_j)=0. \tag{A11}$$

Therefore,

$$0 = f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i) f_{\lambda_{j+1},\lambda_j}(x_{j+1}-x_j)$$
$$= f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{j+1},\lambda_j}(x_{j+1}-x_j). \tag{A12}$$

From Eqs. (A2)–(A12), it follows that $f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)$ makes no difference anywhere in the product $\Pi_{i=1}^{N-1} f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)$. It can therefore be dropped out of every term i. Hence

$$\prod_{i=1}^{N-1} [f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i) f_{\lambda_{N+1},\lambda_i}(x_{N+1}-x_i)] \times f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N)$$
$$= \prod_{i=1}^{N-1} [f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i)] f_{\lambda_{N+1},\lambda_N}(x_{N+1}-x_N)$$
$$= \prod_{i=1}^{N} f_{\lambda_{i+1},\lambda_i}(x_{i+1}-x_i). \tag{A13}$$

Referring back to Eq. (A1), we see that this proves the theorem.

[1] For a review, see I. Balberg, Philos. Mag. B **55**, 991 (1987).

[2] S. A. Safran, I. Webman, and G. S. Grest, Phys. Rev. A **32**, 506 (1985); A. L. R. Bug, S. A. Safran, G. S. Grest, and I. Webman, Phys. Rev. Lett. **55**, 1896 (1985).

[3] A. Drory, Phys. Rev. E **54**, 5992 (1996); **54**, 6003 (1996).

[4] H. Takahashi, Proc. Phys. Math. Soc. Jpn. **24**, 60 (1942) [reprinted in *Mathematical Physics in One Dimension*, edited by E. H. Lieb and D. C. Mattis (Academic, New York, 1966)].

[5] Z. W. Salsburg, R. W. Zwanzig, and J. G. Kirkwood, J. Chem. Phys. **21**, 1098 (1953) [reprinted in *Mathematical Physics in One Dimension* (Ref. [4])].

[6] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[7] E. Cinlar and S. Torquato, J. Stat. Phys. **78**, 827 (1995).